

# $P(\phi)_2$ Quantum Field Theories and Segal's Axioms

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**Abstract:** The purpose of this paper is to show that  $P(\phi)_2$  Euclidean quantum field theories satisfy axioms of the type advocated by Graeme Segal.

## 1. Introduction

Throughout this paper, we fix (a bare mass)  $m_0 > 0$ , and a polynomial  $P : \mathbb{R} \rightarrow \mathbb{R}$  which is bounded from below.

If  $\hat{\Sigma}$  is a closed Riemannian surface, the classical  $P(\phi)_2$ -action is the local functional

$$\mathcal{A} : \mathcal{F}(\hat{\Sigma}) \rightarrow \mathbb{R} : \phi \rightarrow \int_{\hat{\Sigma}} \left( \frac{1}{2} (|d\phi|^2 + m_0^2 \phi^2) + P(\phi) \right) dA, \quad (1)$$

where  $\mathcal{F}(\hat{\Sigma})$  is the appropriate domain of  $\mathbb{R}$ -valued fields on  $\hat{\Sigma}$  for  $\mathcal{A}$ . A heuristic expression for the  $P(\phi)_2$ -Feynmann-Kac measure is

$$\exp(-\mathcal{A}(\phi)) \prod_{x \in \hat{\Sigma}} d\lambda(\phi(x)), \quad (2)$$

where  $d\lambda(\phi(x))$  denotes Lebesgue measure for  $\phi(x) \in \mathbb{R}$ .

It is notoriously difficult to understand the meaning of a generic heuristic Feynmann-Kac expression. Such an expression may not be usefully represented by a measure at all. However, for the  $P(\phi)_2$  action (1), there is a well-known interpretation of (2), as a finite measure on generalized functions,

$$e^{-\int_{\hat{\Sigma}} P(\phi) : C_0} \det_{\zeta} (m_0^2 + \Delta)^{-1/2} d\phi_C, \quad (3)$$

where  $C_0 = -\frac{1}{2\pi} \ln(m_0 d(x, y))$ ,  $C = (m_0^2 + \Delta)^{-1}$ ,  $d\phi_C$  is the Gaussian probability measure with covariance  $C$ ,  $\int : P(\phi) :_{C_0}$  denotes a regularization of the nonlinear interaction, and  $\det_{\zeta}$  denotes the zeta function determinant.

Our main purpose is to show that these Feynmann-Kac measures lead naturally to a theory satisfying a primitive form of Segal's axioms for a quantum field theory: to a circle  $S_R^1$  of radius  $R$ , there is an associated Hilbert space, to a compact Riemannian surface with geodesic boundary components there is an associated operator, and these assignments have functorial properties consistent with heuristic manipulations of path integrals.

The plan of the paper is the following.

In section 2 we introduce some notation used throughout the paper (we largely follow the conventions in [9]). We also recall the primitive form of Segal's axioms, roughly expressed above.

In section 3, and Appendix A, we discuss the  $P(\phi)_2$ -Hilbert spaces. The main point is that for  $P(\phi)_2$  theories, in Segal's framework, the Hilbert space is independent of  $P$ ,  $m_0$ , and the metric on space (a union of circles). Moreover, we can focus on the real part of the Hilbert space, which simplifies matters somewhat. This real Hilbert space is defined in terms of the notion of the space of half-densities associated to a measure class (Appendix A).

To define the vector that corresponds to a Riemannian surface with geodesic boundary, in section 4 we consider the Feynmann-Kac measure which is associated to the double of the surface (following [9] or [15]). The fundamental result, established by constructive field theorists in the 70's, is that (3) is indeed a well-defined finite measure.

In section 5, we show that the Feynmann-Kac measures naturally lead to a representation of Segal's category of compact Riemannian surfaces with geodesic boundaries. The free case ( $P = 0$ ) has been considered previously, and more deeply, by Segal ([13],[14]), and, from a different point of view, by Dimock ([5]). The main technical tool is the work of Burghlelea, Friedlander, and Kappeler on locality properties of zeta function determinants ([3]).

## 2. Preliminaries

Throughout this paper all function spaces are real, and all manifolds are oriented.

Suppose that  $X$  is a closed Riemannian manifold. The test function space is  $\mathcal{D}(X) = C^\infty(X; \mathbb{R})$ , with the Frechet topology of uniform convergence of all derivatives. We will write  $f, g, h, \dots$  for test functions. The space of distributions is  $\mathcal{D}'(X)$ , with the weak topology relative to  $\mathcal{D}(X)$ . The Riemannian volume induces a map with dense image

$$\mathcal{D}(X) \rightarrow \mathcal{D}'(X) : f \rightarrow f dV. \quad (4)$$

We will write  $\phi, \psi, \dots$  for distributions. The pairing of a test function and distribution will be denoted by  $(f, \phi)$ .

The positive Laplacian on functions will be denoted by  $\Delta = \Delta_X$ , and  $C(m, X)$  will denote the operator  $(m^2 + \Delta)^{-d/2}$ , where  $d = \dim(X)$ . In this paper we will only consider  $d = 1, 2$ . We will often abbreviate  $C(m, X)$  to  $C$ , when there is minimal risk of confusion.

The Gaussian probability measure on  $\mathcal{D}'(X)$  with Cameron-Martin Hilbert space

$$W^{d/2}(X, m) = \{\phi : C(m, X)^{-1/2}\phi \in L^2(X, dV)\}, \quad (5)$$

with inner product

$$\langle \phi, \psi \rangle_{W^{d/2}} = \int_X C^{-1/2} \phi C^{-1/2} \psi dV, \quad (6)$$

will be denoted by  $d\phi_{C(m,X)}$ . Heuristically,

$$d\phi_C = d\phi_{C(m,X)} = \frac{1}{Z} e^{-\frac{1}{2} \int_X \phi(m^2 + \Delta_X)^{d/2} \phi dV} d\lambda(\phi), \quad (7)$$

where  $d\lambda(\phi)$  denotes the heuristic Riemannian volume on fields induced by  $dV$ ; rigorously, the Fourier transform is given by

$$\int e^{-i(f,\phi)} d\phi_C = e^{-\frac{1}{2}(f,Cf)}. \quad (8)$$

*Remark 1.* (a) An  $f \in \mathcal{D}(X)$  defines a linear function  $(f, \cdot)$  on  $\mathcal{D}'(X)$ . One has

$$\int |(f, \phi)|^2 d\phi_C = \|f\|_{W^{-d/2}(X,m)}^2. \quad (9)$$

Therefore there is an isometric injection

$$W^{-d/2}(X, m) \rightarrow L^2(d\phi_C) \quad (10)$$

(and this can be extended to an isomorphism of Hilbert spaces

$$\hat{S}(W^{-d/2}(X, m)) \rightarrow L^2(d\phi_C), \quad (11)$$

using normal ordering, where  $\hat{S}(\cdot)$  denotes a Hilbert space completion of the symmetric algebra). Whereas we prefer to parameterize the Gaussian  $d\phi_C$  using the Cameron-Martin Hilbert space  $W^{d/2}(X, m)$ , others prefer to think in terms of a random process indexed by the dual Hilbert space  $W^{-d/2}(X, m)$  (see chapter 1 of [15] for a lucid discussion).

(b) Given  $x \in X$ ,  $\delta_x$  lies just outside of  $W^{-d/2}$ , and hence does not quite define an  $L^2$  random variable. This is one point of view on the main technical difficulty of quantum field theory.

**Lemma 1.** *If  $\rho$  is a positive constant, and  $\rho X$  denotes the space obtained by dilating all distances by  $\rho$ , then*

$$d\phi_{C(m,\rho X)} = d\phi_{C(\rho m,X)}. \quad (12)$$

*Proof.* Let  $d = \dim(X)$  and  $dV_X$  the Riemannian volume for  $X$ . Then  $dV_{\rho X} = \rho^d dV_X$ ,  $\Delta_{\rho X} = \rho^{-2} \Delta_X$ , and the Cameron-Martin norm for  $d\phi_{C(m,\rho X)}$  equals

$$\int_X \phi(m^2 + \rho^{-2} \Delta_X)^{d/2} \phi \rho^d dV_X = \int \phi((\rho m)^2 + |\frac{\partial}{\partial \theta}|^2)^{d/2} \phi dV_X, \quad (13)$$

the Cameron-Martin norm for  $d\phi_{C(\rho m,X)}$ .  $\square$

We will write  $S_R^1$ , rather than  $RS^1$ , to denote  $S^1$  with the metric  $ds = R d\theta$ .

Suppose that  $\Sigma$  is a compact Riemannian surface with boundary,  $S$ . We are assuming that  $S$  has an intrinsic orientation which, at a given point, may or may not agree with the orientation induced by  $\Sigma$ . We define  $W^1(\Sigma, m)$  to consist of  $L^2$  functions with locally  $L^2$ -integrable partial derivatives such that the norm squared

$$\int_{\Sigma} (d\phi \wedge *d\phi + m^2 \phi^2) = \int_{\Sigma} (|d\phi|^2 + m^2 \phi^2) dA < \infty, \quad (14)$$

where  $*$  =  $*_{\Sigma}$  denotes the star operator. This is consistent with (5)-(6), when  $S$  is empty. As a topological space,  $W^1(\Sigma, m)$  is independent of  $m$ . When the specific metric is not needed, we will simply write  $W^1(\Sigma)$ .

Because  $S$  is smooth, smooth functions are dense in  $W^1(\Sigma)$ . The restriction map

$$C^{\infty}(\Sigma) \rightarrow C^{\infty}(S) \quad (15)$$

extends continuously to a map, the trace,

$$W^1(\Sigma) \rightarrow W^{1/2}(S), \quad (16)$$

The trace induces a short exact sequence of topological spaces,

$$0 \rightarrow W_0^1(\Sigma, m) \rightarrow W^1(\Sigma, m) \rightarrow W^{1/2}(S) \rightarrow 0. \quad (17)$$

The orthogonal complement of the kernel is

$$W_0^1(\Sigma, m)^{\perp} = \{\phi \in W^1(\Sigma) : (m^2 + \Delta)\phi = 0 \text{ in } \Sigma \setminus S\}, \quad (18)$$

the solution space of the Helmholtz equation. The quotient Hilbert space structure on  $W^{1/2}(S)$  is defined by a positive first order pseudodifferential operator  $D_{\Sigma}$  on  $S$ . The expression for this operator can be derived from the isomorphism induced by the trace,

$$W_0^1(\Sigma, m)^{\perp} \rightarrow W^{1/2}(S) : \Phi \rightarrow \phi = \Phi|_S. \quad (19)$$

For a smooth solution  $\Phi$  of the Helmholtz equation, using Stokes's theorem,

$$\int_{\Sigma} (d\Phi \wedge *_{\Sigma} d\Phi + m^2 *_{\Sigma} \Phi^2) = \int_{\Sigma} d(\Phi \wedge *_{\Sigma} d\Phi) = \int_{\partial\Sigma} \Phi \wedge *_{\Sigma} d\Phi \quad (20)$$

(here  $\partial\Sigma$  denotes the boundary with induced orientation). Consequently

$$D_{\Sigma}\phi = \pm *_{\Sigma} (*_{\Sigma} d\Phi)|_S, \quad (21)$$

where the sign is positive if the intrinsic and induced orientations agree. When  $S$  is totally geodesic, this is simply the unit outward normal derivative of  $\Phi$  along  $S$ . The operator  $D_{\Sigma}$  is often referred to as the Dirichlet to Neumann operator. The principal symbol of the operator  $D_{\Sigma}^2$  is the induced metric on  $T^*S$  (see subsection 4.4 of [3]).

*2.1. Segal's definition (a primitive version).* As in section 4 of [13], let  $\mathcal{C}_{metric}$  denote the category for which the objects are oriented closed Riemannian 1-manifolds, and the morphisms are oriented compact Riemannian 2-manifolds with totally geodesic boundaries.

**Definition 1.** *A primitive 2-dimensional unitary quantum field theory is a representation of  $\mathcal{C}_{metric}$  by separable Hilbert spaces and Hilbert-Schmidt operators such that disjoint union corresponds to tensor product, orientation reversal corresponds to adjoint,  $\mathcal{C}_{metric}$ -isomorphisms correspond to natural Hilbert space isomorphisms.*

*Remark 2.* (a). The naturality of the isomorphisms has to be spelled out in terms of various commuting diagrams, which we will leave to the reader's imagination (see section 4 of [13] for some additional details).

(b) It is interesting to ask to what extent this definition captures the notion of locality for a qft. Segal has recently advocated additional axioms, which address the following two (apparent) shortcomings: (1) a generic surface does not have many closed geodesics, and in particular a morphism may not be divisible (i.e. expressible as a composition); and (2) a circle can be cut into intervals, and the Hilbert space should be recoverable from data associated to the intervals (see pages 424-425 of [13]).

(c) For a divisible morphism  $\Sigma : S \rightarrow S$ , the definition implies that the corresponding operator is trace class. In this case it also follows that the trace equals the partition function of the closed surface obtained by sewing along  $S$ .

To show that  $P(\phi)_2$  satisfies this primitive form of Segal's axioms, we will do the following.

To  $S_R^1$  we will associate a real Hilbert space, which we will ultimately denote by  $\mathcal{H}(S^1)$ , because this space will not depend on  $R$ ,  $P$ , or  $m_0$ . This space will carry a natural  $Rot(S^1)$  action. Since disjoint union of circles corresponds to tensor product of Hilbert spaces, and a connected oriented Riemannian 1-manifold is isomorphic to  $S_R^1$ , for a uniquely determined  $R$ , where the isomorphism is determined up to a rotation, this determines the Hilbert space for more general 1-manifolds. Since we will work with real Hilbert spaces, we will not have to explicitly keep track of duals.

Let  $\Sigma$  denote an oriented compact Riemannian surface with geodesic and arclength parameterized boundary components. A component of  $\partial\Sigma$  is said to be outgoing if the parameterization agrees with the induced orientation, and ingoing otherwise. The union of outgoing boundary components will be denoted by  $(\partial\Sigma)_{out}$ , and the union of ingoing boundary components will be denoted by  $(\partial\Sigma)_{in}$ . To this surface we will associate a trace class operator

$$\mathcal{Z}(\Sigma) : \mathcal{H}((\partial\Sigma)_{in}) \rightarrow \mathcal{H}((\partial\Sigma)_{out}). \quad (22)$$

Let  $|\Sigma|$  denote the morphism obtained from  $\Sigma$  by reversing the orientation of all incoming circles. Because the Hilbert spaces we consider are real, so that we can identify such a space with its dual, there are equalities

$$\mathcal{Z}(\Sigma) = \mathcal{Z}(|\Sigma|) \in \mathcal{H}(\partial\Sigma) = \mathcal{H}(\partial|\Sigma|). \quad (23)$$

Suppose that  $\Sigma_1$  and  $\Sigma_2$  are two such surfaces, and the number of outgoing boundary components of  $\Sigma_1$  is the same as the number of ingoing boundary

components of  $\Sigma_2$ . We can glue these Riemannian manifolds along  $(\partial\Sigma_1)_{out}$  and  $(\partial\Sigma_2)_{in}$  to obtain another such surface  $\Sigma_2 \circ \Sigma_1$ . We will show

$$\mathcal{Z}(\Sigma_2 \circ \Sigma_1) = \mathcal{Z}(\Sigma_2) \circ \mathcal{Z}(\Sigma_1). \quad (24)$$

### 3. The Hilbert Space $\mathcal{H}(S^1)$ .

To define the Hilbert space, we will use the notion of the space of half-densities of a measure class. This is described in Appendix A.

Suppose that  $M > 0$ . For all of the  $P(\phi)_2$  theories,

$$\mathcal{H}(S_R^1) = \mathcal{H}(\mathcal{C}(M, S_R^1)) \quad (25)$$

where  $\mathcal{C}(M, S_R^1)$  is the measure class on  $\mathcal{D}'(S^1)$  represented by the probability measure  $d\phi_{C(M, S_R^1)}$  on  $\mathcal{D}'(S^1)$ .

We also want to allow the possibility that  $M = 0$ . This is the nonfinite measure

$$d\phi_{C(0, S_R^1)} = \lim_{M \downarrow 0} \frac{\sqrt{2\pi}}{M} d\phi_{C(M, R)}. \quad (26)$$

A real generalized function on  $S^1$  has a Fourier series

$$\phi = \phi_0 + \sum_1^\infty (\phi_n e^{in\theta} + \bar{\phi}_n e^{-in\theta}) \quad (27)$$

In these coordinates, if  $M > 0$ ,  $d\phi_{C(M, S_R^1)}$  is the infinite product of probability measures

$$d\phi_{C(M, S_R^1)} = \frac{M}{\sqrt{2\pi}} e^{-\frac{1}{2}M^2\phi_0^2} d\lambda(\phi_0) \prod_{n=1}^\infty d\mu_n^{(MR)} \quad (28)$$

where

$$d\mu_n^{(M)} = \frac{(M^2 + n^2)^{1/2}}{2\pi} e^{-\frac{1}{2}(M^2 + n^2)^{1/2}|\phi_n|^2} d\lambda(\phi_n) \quad (29)$$

If  $M = 0$ , then

$$d\phi_{C(0, S^1)} = d\lambda(\phi_0) \prod_{n=1}^\infty d\mu_n^{(0)}. \quad (30)$$

Note there is no dependence on  $R$  when  $M = 0$ .

**Lemma 2.** *The measure class  $\mathcal{C}(M, S_R^1)$  is independent of  $M \geq 0$  and  $R$ .*

*Proof.* In addressing this question, we can ignore the  $\phi_0$  factor.

Kakutani's theorem (Theorem 2.12.7, page 92, of [1]), asserts that the two infinite product measures

$$\prod d\mu_n^{(mr)} \quad \text{and} \quad \prod d\mu_n^{(MR)} \quad (31)$$

are either equivalent or disjoint, and they are equivalent if and only if the inner product between the corresponding positive half-densities is positive, i.e.

$$\prod_{n=1}^\infty \int \sqrt{d\mu_n^{(mr)} d\mu_n^{(MR)}} > 0. \quad (32)$$

In doing this calculation, we can clearly assume  $r = R = 1$ .

The  $n^{th}$  factor of (32) equals

$$\frac{(M^2 + n^2)^{1/4}(m^2 + n^2)^{1/4}}{2\pi} \int_{\mathbb{C}} e^{-\frac{1}{4}((M^2+n^2)^{1/2}+(m^2+n^2)^{1/2})|x_n|^2} d\lambda(x_n) \quad (33)$$

$$= n(1 + \frac{M^2}{n^2})^{1/2}(1 + \frac{m^2}{n^2})^{1/2} \frac{2}{(M^2 + n^2)^{1/2} + (m^2 + n^2)^{1/2}} \quad (34)$$

$$= (1 + \frac{M^2}{n^2})^{1/2}(1 + \frac{m^2}{n^2})^{1/2} \frac{2}{(1 + \frac{M^2}{n^2})^{1/2} + (1 + \frac{m^2}{n^2})^{1/2}} \quad (35)$$

This has a positive infinite product over  $n$ .  $\square$

We will need a more sophisticated result along these same lines. Suppose that  $D$  is a positive classical pseudodifferential operator of order 1 on  $S$ , a compact connected one-manifold (e.g.  $D = (M^2 + \Delta_{S^1})^{1/2}$ ). The principal symbol of the operator  $D^2$  determines a Riemannian metric on  $S$ , hence a radius  $R$ . By choosing an arclength coordinate  $R\theta$ , we can suppose  $S = S^1$  and the metric is  $Rd\theta$ .

**Proposition 1.** *Let  $D_1$  and  $D_2$  denote two operators as above such that  $D_1$  and  $D_2$  have the same principal symbols. Let  $Rd\theta$  denote the corresponding metric. Then the Gaussian measures  $\mu_i$  with Cameron-Martin inner products*

$$\langle \phi, \psi \rangle_i = \int_S \phi D_i \psi Rd\theta \quad (36)$$

*are equivalent.*

*Proof.* Obviously

$$\langle \phi, \psi \rangle_2 = \langle D_1^{-1} D_2 \phi, \psi \rangle_1 \quad (37)$$

Because  $D_1$  and  $D_2$  are classical pseudodifferential operators, and they have the same principal symbols,

$$D_1^{-1} D_2 = 1 + A \quad (38)$$

where  $A$  is a pseudodifferential operator of order  $-1$ . Because  $S$  is one dimensional,  $A$  is Hilbert-Schmidt. This implies that the  $\mu_i$  are equivalent (see Theorem 6.3.2, page 286, of [1], or Theorem I.23, page 41, of [15]).  $\square$

Since the Hilbert space corresponding to a circle is independent of  $R$ ,  $M$ , and  $P$ , we will denote it simply by  $\mathcal{H}(S^1)$ . More generally, given a closed 1-manifold  $S$ , there is a measure class associated to  $W^{1/2}(S)$ , and we will denote the associated real Hilbert space of half-densities by  $\mathcal{H}(S)$ . This space is intrinsic to  $S$ , and it is naturally isomorphic to the tensor product of the  $\mathcal{H}(S_i)$ , where the  $S_i$  (ordered in some way) denote the connected components of  $S$ ; see (5) of Appendix A.

#### 4. Feynmann-Kac Measures

To define the trace class operators corresponding to surfaces, we will need a number of technical results about Feynmann-Kac measures for closed Riemannian surfaces.

Suppose that  $\hat{\Sigma}$  is a closed oriented Riemannian surface. Let  $\{f_k\}$  denote an orthonormal basis of real eigenfunctions for the positive Laplace operator,  $\Delta$ , where  $\Delta f_k = \lambda_k f_k$ ,  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ . A generalized function on  $\hat{\Sigma}$  has an expansion

$$\phi = \sum \phi_k f_k = \phi_0 + \psi. \quad (39)$$

In the coordinates  $\phi_k \in \mathbb{R}$ ,  $d\phi_{C(M, \hat{\Sigma})}$  is the infinite product measure

$$d\phi_{C(M, \hat{\Sigma})} = \prod d\mu_n^{(M)} \quad (40)$$

where

$$d\mu_n^{(M)} = \sqrt{\frac{M^2 + \lambda_k}{2\pi}} e^{-\frac{(M^2 + \lambda_k)}{2} \phi_k^2} d\phi_k \quad (41)$$

We also define

$$d\phi_{C(0, \hat{\Sigma})} = d\lambda(\phi_0) \times d\psi_{C(0, \hat{\Sigma})} = d\lambda(\phi_0) \prod_{k=1}^{\infty} \sqrt{\frac{\lambda_k}{2\pi}} e^{-\frac{\lambda_k}{2} \phi_k^2} d\phi_k \quad (42)$$

*Remark 3.* (a) The measure  $d\psi_{C(0, \hat{\Sigma})}$  is a Gaussian measure on  $\mathcal{D}'(\hat{\Sigma})_0$ , where  $\psi \in \mathcal{D}'_0$  means  $\psi(1) = 0$  ( $\mathcal{D}'_0$  is the dual of  $\mathcal{D}/\mathbb{R}$ ), and the Cameron-Martin inner product is

$$\int_{\hat{\Sigma}} d\psi_1 \wedge *d\psi_2. \quad (43)$$

(b) The space  $\mathcal{D}'_0$  depends on the  $C^\infty$  structure of  $\hat{\Sigma}$  (diffeomorphisms act naturally on  $\mathcal{D}/\mathbb{R}$ , and hence its dual). The Cameron-Martin inner product depends on the conformal structure of  $\hat{\Sigma}$  (because it involves the star operator on one-forms). The decomposition of distributions

$$\mathcal{D}'(\hat{\Sigma}) = \mathbb{R} \oplus \mathcal{D}'(\hat{\Sigma})_0 : \phi = \phi_0 + \psi, \quad (44)$$

as in (39), depends on the volume element of  $\hat{\Sigma}$  (so that  $\phi_0$  can be interpreted as a distribution). Consequently the measure  $d\phi_{C(0, \hat{\Sigma})}$  depends on the Riemannian structure of  $\hat{\Sigma}$ ; the measure  $d\psi_{C(0, \hat{\Sigma})}$  is conformally invariant.

Let  $\mathcal{C}(M, \hat{\Sigma})$  denote the measure class of  $d\phi_{C(M, \hat{\Sigma})}$ .

**Lemma 3.** *For constant  $\rho > 0$ , the measure class  $\mathcal{C}(M, \rho \hat{\Sigma})$  is independent of  $M \geq 0$  and  $\rho$ .*



*Proof.* The independence of  $M$  is essentially the same as for Lemma 2. The point is that  $\lambda_k$  is asymptotically  $k$ .

We again apply Kakutani's criterion for equivalence, as in (32). We can also ignore the zero mode,  $\phi_0$ .

The  $n^{\text{th}}$  factor,  $\int \sqrt{d\mu_n^{(M)} d\mu_n^{(m)}}$  equals

$$\frac{(M^2 + \lambda_n)^{1/4} (m^2 + \lambda_n)^{1/4}}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-\frac{1}{4}(M^2 + \lambda_n + m^2 + \lambda_n)|\phi_n|^2} d\lambda(\phi_n) \quad (45)$$

$$= \left(\frac{\lambda_n}{2\pi}\right)^{1/2} \left(1 + \frac{M^2}{\lambda_n}\right)^{1/4} \left(1 + \frac{m^2}{\lambda_n}\right)^{1/4} \left(\frac{4\pi}{2\lambda_n + M^2 + m^2}\right)^{1/2} \quad (46)$$

$$= \left(1 + \frac{M^2}{\lambda_n}\right)^{1/4} \left(1 + \frac{m^2}{\lambda_n}\right)^{1/4} \left(1 + \frac{M^2 + m^2}{2\lambda_n}\right)^{-1/2} \quad (47)$$

$$= 1 + O\left(\frac{1}{\lambda_n^2}\right) \quad (48)$$

Thus the inner product is positive, and the measures are equivalent.

This proves the independence of  $M$ . The independence of  $\rho$  now follows from Lemma 1.  $\square$

*Remark 4.* If  $\hat{\Sigma}$  is replaced by a manifold of dimension  $d$ , and we consider an action defined by a second order operator, then independence of mass holds if and only if  $d < 4$ , because  $\lambda_n$  is asymptotic to  $n^{2/d}$ .

In the formulation of the following Lemma, we will use a basic fact, due to Colella and Lanford, about the free field  $d\phi_{C(M, \hat{\Sigma})}$ . This will be used frequently in the remainder of the paper. A typical configuration  $\phi$  for the free field is not an ordinary function (or even a signed measure). However, given a nice foliation of  $\hat{\Sigma}$  by 1-submanifolds, a typical configuration can be thought of as a continuous function (of a transverse parameter) with values in distributions along the leaves. A precise formulation of this, in the case of  $\mathbb{R}^2$ , can be found in [4] (Theorem 1.1, part (b), page 45, and see the paragraph following the Theorem, for further comment).

**Lemma 4.** *Suppose that  $c : S_R^1 \rightarrow \hat{\Sigma}$  is an isometric embedding. Then the projection of  $d\phi_{C(M, \hat{\Sigma})}$  to a measure on  $\mathcal{D}'(S^1)$  belongs to the measure class  $\mathcal{C}(M, S_R^1)$ .*

*Proof.* Since  $d\phi_{C(M, \hat{\Sigma})}$  is a Gaussian measure, its projection must be a Gaussian measure. One way to calculate the image of a Gaussian is to consider the map of Cameron-Martin spaces, which in this case is the trace map

$$W^1(\hat{\Sigma}, M) \rightarrow W^{1/2}(S^1), \quad (49)$$

where the inner product on the target is determined by a positive first order pseudodifferential operator  $D$ , obtained by considering the  $W^1$  inner product on Helmholtz solutions on  $\hat{\Sigma} \setminus c(S^1)$ , as in (17)-(21).

To relate this directly to (17)-(21), cut the closed surface  $\hat{\Sigma}$  along  $c$  to obtain a compact surface  $\Sigma$  with two boundary components, one of which is positively parameterized by  $c$ , and one of which is negatively parameterized by  $c$ . This

induces a pseudodifferential operator  $D_\Sigma$  on  $\partial\Sigma$ , as in (17)-(21). This yields two pseudodifferential operators  $D_\pm$  on  $S^1$ , corresponding to the positive and negative  $c$ -parameterizations. The operator  $D = D_+ + D_-$ , by (21).

Thus  $D^2$  has principal symbol which is proportional to the induced metric on  $T^*S^1$ , and the Lemma follows from Proposition 1 (recall also that the measure class  $\mathcal{C}(M, S_R^1)$  is independent of  $M$  and  $R$ , by Lemma 2).

Alternatively, if  $C(x, y)$  denotes the kernel for  $\mathcal{C}(M, \hat{\Sigma})$ , the covariance for the projection is given by

$$C'(\theta, \theta') = C(c(\theta), c(\theta')). \quad (50)$$

One can read off the principal symbols for  $C'$  and its inverse  $D$  from the fact that  $C$  is asymptotically  $-\frac{1}{2\pi}\ln(Md(x, y))$ , as  $d(x, y) \rightarrow 0$ .  $\square$

*4.1. Normal Ordering.* From now on we will use our fixed bare mass  $m_0 > 0$ . Let  $C = C(m_0, \hat{\Sigma})$ . If  $f \in \mathcal{D}(\hat{\Sigma})$ , then  $(f, \cdot) \in L^2(d\phi_C)$ . By definition

$$: (f, \cdot)^n :_C = H_n^{(f, Cf)}((f, \cdot)) \in L^2(d\phi_C), \quad (51)$$

where  $H_n^\alpha$  denotes the  $n$ th Hermite polynomial for the Gaussian  $(2\pi\alpha)^{-1/2}e^{-\frac{1}{2\alpha}x^2}d\lambda(x)$  (there are a number of different ways to motivate this definition; see either section 6.3 of [9] or chapter 1 of [15]). For example

$$: (f, \cdot)^4 :_C = (f, \cdot)^4 - 6(f, Cf)(f, \cdot)^2 + 3(f, Cf)^2 \quad (52)$$

One can define  $: (f, \cdot)^n :_C$  equally well for  $f \in W^{-1}(\hat{\Sigma})$ , because of (10). Unfortunately, given a point  $x \in \hat{\Sigma}$ ,  $\delta_x$  is not in  $W^{-1}$ , and in fact it is impossible to define  $(\delta_x, \cdot)$  as a random variable with respect to  $d\phi_{C(m_0, \hat{\Sigma})}$  (the support of this measure consists of genuine distributions). However, for  $n \geq 0$ , it is possible to define a regularization  $: (\delta_x, \cdot)^n :_C$ , as a distribution; that is, given  $\rho \in \mathcal{D}(\hat{\Sigma})$ ,

$$\int_{\hat{\Sigma}} : (\delta_x, \cdot)^n :_C \rho(x) dA(x) \quad (53)$$

is a well-defined integrable random variable with respect to  $d\phi_C$ . For example (see section 8.5, page 152, of [9]),

$$: (\delta_x, \cdot)^4 :_C = \lim_{t \downarrow 0} [(\delta_{t,x}, \cdot)^4 - 6(\delta_{t,x}, C\delta_{t,x})(\delta_{t,x}, \cdot)^2 + 3(\delta_{t,x}, C\delta_{t,x})^2] \quad (54)$$

where  $\delta_{t,x} \in \mathcal{D}(\hat{\Sigma})$  satisfies  $\delta_{t,x} \rightarrow \delta_x$  as  $t \downarrow 0$ . We will always choose the functions  $\delta_{t,x}$  to have compact support which shrinks to  $x$ , and for these functions to depend smoothly on  $x$ .

Now suppose that we think of  $C$  as a kernel function (which we can do because we have a Riemannian background, and in particular an area form). A fundamental fact is that, near the diagonal,

$$C(m_0, \hat{\Sigma})(x, y) = C_0(m_0, x, y) + C_f(m_0, x, y), \quad (55)$$

where  $C_0(m_0, x, y) = -\frac{1}{2\pi}\ln(m_0d(x, y))$  and  $C_f$  is smooth. We will often suppress the argument  $m_0$ .

For  $\rho \in \mathcal{D}(\hat{\Sigma})$ , we define

$$\int : (\delta_x, \cdot)^n :_{C_0} \rho(x) dA(x) = \lim_{t \downarrow 0} \int H_n^{(\delta_{t,x}, C_0 \delta_{t,x})}((\delta_{t,x}, \cdot)) \rho(x) dA(x) \quad (56)$$

For example

$$: (\delta_x, \cdot)^4 :_{C_0} = \lim_{t \downarrow 0} [(\delta_{t,x}, \cdot)^4 - 6(\delta_{t,x}, C_0 \delta_{t,x})(\delta_{t,x}, \cdot)^2 + 3(\delta_{t,x}, C_0 \delta_{t,x})^2] \quad (57)$$

*Remark 5.* This is local: the calculation of  $(\delta_{t,x}, C_0 \delta_{t,x})$  depends on arbitrarily small neighborhoods of  $x$  as  $t \downarrow 0$ . In a first version of this paper, I claimed that one could just as well use  $C$ . But in general this is false, because for fixed  $x$ , there is a constant in the asymptotic expansion of  $C$  (the value  $C_f(x, x)$ ), which is not zero, and which is not locally determined.

One can also express (56) in terms of regularization by  $C$ : by a standard formula for ‘finite change of Wick order’ (see (8.6.1) of [9]), (56) equals

$$\sum_{j=0}^{[n/2]} \frac{n!}{(n-2j)!j!2^j} \int C_f(x, x)^j : (\delta_x, \cdot)^{n-2j} :_C \rho(x) dA(x) \quad (58)$$

For example

$$: (\delta_x, \cdot)^4 :_{C_0} = : (\delta_x, \cdot)^4 :_C + 6C_f(x, x) : (\delta_x, \cdot)^2 :_C + 3C_f(x, x)^2. \quad (59)$$

The important point is that these regularizations agree up to lower order terms.

In general we define  $: P((\delta_x, \cdot)) :_{C_0}$  by linear extension. We will occasionally abbreviate this simply to  $: P :_{C_0}$ , or, if we need to display the argument, to  $: P(\phi) :_{C_0}$  [rather than the more cumbersome  $: P((\delta_x, \cdot)) :_{C_0}(\phi)$ ].

The following is one of the fundamental results of constructive quantum field theory.

**Theorem 1.** *Suppose that  $P(\phi)$  is bounded below. Then  $\exp(-\int_{\hat{\Sigma}} : P :_{C_0}) \in L^1(d\phi_{C(m_0, \hat{\Sigma})})$ .*

This follows, with relatively minor modifications, from the arguments in section 8.6 of [9], or V.2 of [15] (Note that a closed Riemannian surface is conformally equivalent to a constant curvature surface, and hence by uniformization can be presented as a nice bounded region with generalized periodic boundary conditions, and conformally Euclidean metric - with the exception of the sphere).

**Definition 2.** *The Feynmann-Kac measure for  $\hat{\Sigma}$  is the finite measure on  $\mathcal{D}'(\hat{\Sigma})$*

$$e^{-\int_{\hat{\Sigma}} : P :_{C_0} dA(x)} \det_{\zeta}(\Delta + m_0^2)^{-1/2} d\phi_C. \quad (60)$$

At a heuristic level, we can say that the  $\zeta$ -determinant is essential because we have (for no good reason) normalized the free background  $d\phi_C$  to have unit mass; we have to add back in the Gaussian volume of the Cameron-Martin space.

## 5. Surfaces, Operators, and Sewing

Suppose that  $\Sigma$  is a compact oriented Riemannian surface, with geodesic and geodesically parameterized boundary components. We also initially assume that all of the boundary components are outgoing, i.e  $\Sigma = |\Sigma|$ . We consider the closed Riemannian surface

$$\hat{\Sigma} = \Sigma^* \circ \Sigma, \quad (61)$$

where  $\Sigma^*$  is the surface obtained by reversing the orientation of everything. Of fundamental importance is the existence of a reflection symmetry through  $\partial\Sigma$ .

Let  $S$  denote  $\partial\Sigma$ , and  $C = C(m_0, \hat{\Sigma})$ . We will write

$$S_*(e^{-\int_{\Sigma} P: c_0 dA} d\phi_C) \quad (62)$$

for the projection of this measure to a finite measure on  $\mathcal{D}'(S)$ , which exists by Lemma 4.

**Definition 3.** For  $\Sigma$  as above, we define

$$\mathcal{Z}_1(\Sigma) = \mathcal{Z}_1(|\Sigma|) = (S_*(e^{-\int_{\Sigma} P: c_0 dA} d\phi_C))^{1/2} \in \mathcal{H}(S), \quad (63)$$

and

$$\mathcal{Z}(\Sigma) = \det_{\zeta}(\Delta_{\hat{\Sigma}} + m_0^2)^{-1/4} \mathcal{Z}_1(\Sigma) \in \mathcal{H}(S). \quad (64)$$

For a closed surface  $\hat{\Sigma}$ , we define  $\mathcal{Z}(\hat{\Sigma})$  to be the integral of its Feynmann-Kac measure.

Note that for a morphism  $\Sigma : S_1 \rightarrow S_2$ , it follows immediately from this definition that  $\mathcal{Z}(\Sigma)$  represents a Hilbert-Schmidt operator.

**Theorem 2.** Suppose that  $\Sigma_1$  and  $\Sigma_2$  are two morphisms which can be composed. Then

$$\mathcal{Z}(\Sigma_3) = \mathcal{Z}(\Sigma_2) \circ \mathcal{Z}(\Sigma_1), \quad (65)$$

where  $\Sigma_3 = \Sigma_2 \circ \Sigma_1$ .

(b) Suppose  $\Sigma : S \rightarrow S$  is divisible. Then  $\mathcal{Z}(\Sigma)$  is trace class, and

$$\text{trace}(\mathcal{Z}(\Sigma)) = \mathcal{Z}(\hat{\Sigma}), \quad (66)$$

where  $\hat{\Sigma}$  is the closed surface obtained by gluing  $\Sigma$  to itself along  $S$ .

The rest of this section is devoted to the proof of this Theorem. For (a) there are three possibilities: both of  $(\partial\Sigma_1)_{in}$  and  $(\partial\Sigma_2)_{out}$  are empty, one is empty, and neither is empty. The line of argument for each of these cases is exactly the same, but the notational details vary. We will carry out all the details for the second possibility.

There are basically four parts to the argument. In the first part, we study the disintegration of the free Feynmann-Kac measure with respect to its projection to a measure on generalized functions on the boundary. The second part involves the local character of the nonlinear interaction. The third and fourth parts are tightly intertwined: these parts concern the sewing properties for the normalized background Gaussian measures, and the  $\zeta$ -regularized Gaussian volumes, respectively.

*5.1. Part 1. Decomposition of free backgrounds relative to traces.* As above, we initially suppose that  $\Sigma = |\Sigma|$ , with outgoing boundary  $S$ . The trace map

$$W^1(\hat{\Sigma}) \rightarrow W^{1/2}(S) : \hat{\phi} \rightarrow \hat{\phi}_S \quad (67)$$

corresponds to a Hilbert space decomposition

$$W^1(\hat{\Sigma}, m_0) = W_0^1(\hat{\Sigma}, m_0) \oplus W_0^1(\hat{\Sigma}, m_0)^\perp. \quad (68)$$

In turn,

$$W_0^1(\hat{\Sigma}, m_0) = W_0^1(\Sigma, m_0) \oplus W_0^1(\Sigma^*, m_0) \quad (69)$$

and

$$W_0^1(\hat{\Sigma}, m_0)^\perp = W^1(\hat{\Sigma}, m_0) \cap \ker(\Delta + m_0^2)|_{\hat{\Sigma} \setminus S}. \quad (70)$$

The latter space has two other realizations. On the one hand it is essentially isomorphic to

$$W^1(\Sigma, m_0) \cap \ker(\Delta + m_0^2)|_{\Sigma \setminus S}, \quad (71)$$

because a  $W^1$ -solution  $\hat{\phi}_s$  of the Helmholtz equation on  $\hat{\Sigma} \setminus S$  is necessarily even, i.e. invariant with respect to the mirror symmetry of  $\hat{\Sigma}$  through  $S$  (the even and odd parts of a Helmholtz solution would also be solutions; the odd part vanishes on  $S$ , hence it must be identically zero); hence  $\hat{\phi}_s$  is determined by its restriction to  $\Sigma$ , which we denote by  $\phi_s$ . On the other hand it is also isomorphic to  $W^{1/2}(S)$ , with the inner product determined by  $2D_\Sigma$ , as in (21).

We now want to apply these Hilbert space decompositions to obtain decompositions of the corresponding Gaussian measures, in particular our background Gaussian measures. In the following we will have to distinguish, for example, between  $\hat{\phi} \in W^1(\hat{\Sigma}, m_0)$ , and a typical  $\hat{\phi}$  in the support of  $d\hat{\phi}_C$ ; we will refer to the latter as a random field (rather than introducing some additional notation). We will also implicitly invoke the theorem of Collella-Lansford, which, for example, allows us to make sense of the restriction of a random  $\hat{\phi}$  to  $\Sigma$  or  $S$ .

The Gaussian measure  $d\hat{\phi}_{C(m_0, \hat{\Sigma})}$  has a disintegration relative to its projection to fields on  $S$ :

$$d\hat{\phi}_{C(m_0, \hat{\Sigma})} = \int [d\hat{\phi}_{C(m_0, \hat{\Sigma})} | \hat{\phi}_S = \phi_1] d(S_*(d\hat{\phi}_{C(m_0, \hat{\Sigma})})(\phi_1). \quad (72)$$

The existence of this disintegration is a general fact (Proposition 13, section 2, No. 7, of [2]). But as we will explain in the following paragraphs, the ‘normalized conditional measure’  $[d\hat{\phi}_C | \hat{\phi}_S = \phi_1]$  is a Gaussian probability measure centered at (a classical solution corresponding to)  $\phi_1$ .

The Hilbert space decompositions (68) and (69), and the isomorphism (70), imply that a sample field for the Gaussian  $d\hat{\phi}_{C(m_0, \hat{\Sigma})}$  can be uniquely decomposed as a sum of independent terms:

$$\hat{\phi} = \hat{\phi}_0 + \hat{\phi}_s = \phi_0 + \hat{\phi}_s + \phi_0^*, \quad (73)$$

where  $\phi_0$  ( $\phi_0^*$ , respectively) is a generalized function which is supported on  $\Sigma$  ( $\Sigma^*$ , respectively) and vanishing on  $S$ , and  $\hat{\phi}_s$  is a solution of the Helmholtz equation in  $\hat{\Sigma} \setminus S$ , and determined by its (distributional) boundary value  $\phi_1$  on

$S$ . We will write  $\phi = \phi_0 + \phi_s$  ( $\phi^* = \phi_0^* + \phi_s^*$ , respectively) for the restriction of a random  $\hat{\phi}$  to  $\Sigma$  ( $\Sigma^*$ , respectively).

In particular for *a.e.*  $\phi_1$ , the  $\phi_1$  (normalized) conditioned measure in (72) is a direct product

$$[d\hat{\phi}_{C(m_0, \hat{\Sigma})}|\hat{\phi}|_S = \phi_1] = \quad (74)$$

$$[d\phi_{C(m_0, \Sigma)}|\phi_S = \phi_1] \times [d\phi_{C(m_0, \Sigma^*)}^*|\phi_S^* = \phi_1] \quad (75)$$

*Remark 6.* (a) A random  $\phi$  for  $[d\phi_{C(m_0, \Sigma)}|\phi_S = \phi_1]$  is of the form  $\phi = \phi_0 + \phi_s$ ,  $\phi_0$  is a normalized Gaussian with Cameron-Martin space  $W_0^1(m_0, \Sigma)$ , and  $(\phi_s)_S = \phi_1$ . Thus we could also write

$$[d\phi_{C(m_0, \Sigma)}|\phi_S = \phi_1] = d\phi_{C(m_0, \Sigma, D)}(\phi - \phi_s) \quad (76)$$

where  $C(m_0, \Sigma, D)$  is the inverse of  $m_0^2 + \Delta_\Sigma$  with Dirichlet boundary condition. The right hand side is defined for all solutions  $\phi_s$  of the Helmholtz equation in  $\Sigma \setminus S$ . If one considers a collar  $\{0 \leq t \leq \delta\} \times S^1 \subset \Sigma$  for a boundary component ( $\{t = 0\}$ ), then the Collela-Lanford theorem says that for any  $\epsilon > 0$ , with probability one,  $\phi_0$  is a continuous function of  $t$  with values in  $W^{-\epsilon}(S^1)$ , which vanishes when  $t = 0$ .

(b) To this point we have not given an independent meaning to  $C(m_0, \Sigma)$  or  $d\phi_{C(m_0, \Sigma)}$ . The measure  $d\phi_{C(m_0, \Sigma)}$  can be understood as the Gaussian with Cameron-Martin space  $W^1(\Sigma, m_0)$  (see (14)); a random  $\phi$  is a restriction of a random  $\hat{\phi}$  to  $\Sigma$ . However ' $C(m_0, \Sigma) = (m_0^2 + \Delta)^{-1}$ ' does not have an independent meaning, in reference to  $\Sigma$  alone (because we are interested in a free boundary condition, which is why we introduce the double of  $\Sigma$ ).

In terms of this notation, and using reflection symmetry through  $S$ , we obtain the following

**Lemma 5.** *The pushforward measure  $\mathcal{Z}_1(\Sigma)^2$  (see Definition 3), is given by*

$$\left( \int_{\hat{\phi}_0} e^{-\int_{\hat{\Sigma}} P(\hat{\phi}) : c_0} [d\hat{\phi}_{C(m_0, \hat{\Sigma})}|\hat{\phi}|_S = \phi_1] \right) d(S_* d\hat{\phi}_{C(m_0, \hat{\Sigma})})(\phi_1) \quad (77)$$

$$= \left( \int e^{-\int_{\Sigma} P(\phi) : c_0} [d\phi_{C(m_0, \Sigma)}|\phi|_S = \phi_1] \right)^2 d(S_* d\hat{\phi}_{C(m_0, \hat{\Sigma})})(\phi_1), \quad (78)$$

We now turn to the setup of the theorem.

Suppose that we are given  $\Sigma_1$  and  $\Sigma_2$ . We first suppose that  $\Sigma_1$  has empty incoming boundary, and  $\Sigma_2$  has nonempty outgoing boundary. Thus  $\Sigma_3 = \Sigma_2 \circ \Sigma_1$  also has empty incoming boundary.

Let  $S_1$  denote the outgoing boundary of  $\Sigma_1$  (which is the same as the incoming boundary for  $\Sigma_2$ ), and let  $S_2$  denote the outgoing boundary for  $\Sigma_2$ . We will write  $\phi$  for a field on  $\Sigma_1$ . This field has a decomposition  $\phi = \phi_0 + \phi_s$ , where  $\phi_0$  is Gaussian and  $\phi_s$  is a solution of the Helmholtz equation and determined by the boundary value  $\phi_{S_1}$ . We will similarly write  $\psi$  for a field on  $\Sigma_2$ , with decomposition  $\psi = \psi_0 + \psi_s$ . We will also write  $\phi_i$  for a field on  $S_i$ , and  $C_i$  will denote the covariance  $(m_0^2 + \Delta)^{-1}$  associated to  $|\hat{\Sigma}_i|$ .

For  $\Sigma_3 = \Sigma_2 \circ \Sigma_1$ , there is a finer decomposition, corresponding to the trace map

$$W^1(\Sigma_3) \rightarrow W^{1/2}(S_1) \oplus W^{1/2}(S_2) \quad (79)$$

and the isomorphism

$$W_0^1(\Sigma_3, m_0) = W_0^1(\Sigma_1, m_0) \oplus W_0^1(\Sigma_2, m_0). \quad (80)$$

A random field  $\Phi$  on  $\Sigma_3$  with distribution  $d\Phi_{C_3}$  can be written as a sum of independent Gaussians

$$\Phi = \phi_0 + \psi_0 + \Phi^s, \quad \Phi^s = \phi_s \sqcup \psi_s \quad (81)$$

where  $\phi_s$  and  $\psi_s$  (are random Helmholtz solutions, as before, and) have common boundary value  $\phi_1$  on  $S_1$ ,  $\psi_s$  has boundary value  $\phi_2$  on  $S_2$ , and the  $d\Phi_{C_3}$ -distribution for  $\Phi^s$ , in the coordinates  $(\phi_1, \phi_2)$ , is a Gaussian measure with covariance  $(m_0^2 + \Delta_{\Sigma_3})^{-1}$  restricted to  $S_1 \cup S_2$ . We will write the  $d\Phi_{C_3}$  distribution for  $\Phi^s$  as  $d\Phi_{C_3}^s$ .

*5.2. Part 2. Locality of nonlinear interactions.* We now want to calculate

$$\int_{\phi_1 \in \mathcal{D}'(S_1)} \mathcal{Z}_1(\Sigma_2) \mathcal{Z}_1(\Sigma_1) \quad (82)$$

where the integral is over the common boundary value  $\phi_1 = \phi_{S_1} = \psi_{S_1}$ . By Lemma 5 this

$$= \int_{\phi_1} \left( \int e^{-\int_{\Sigma_1} :P(\phi):_{C_0}} [d\phi_{C_1} | \phi_{S_1} = \phi_1] \right) (S_{1*}(d\hat{\phi}_{C_1}))^{1/2}(\phi_1) \quad (83)$$

$$\left( \int e^{-\int_{\Sigma_2} :P(\psi):_{C_0}} [d\psi_{C_2} | \psi_{S_i} = \phi_i] \right) ((S_1 \sqcup S_2)_*(d\hat{\psi}_{C_2}))^{1/2}(\phi_1, \phi_2) \quad (84)$$

**Proposition 2.** *For a random field  $\Phi$  as in (81),*

$$\int_{\Sigma_1} :P(\phi):_{C_0} + \int_{\Sigma_2} :P(\psi):_{C_0} = \int_{\Sigma_2 \circ \Sigma_1} :P(\Phi):_{C_0}, \quad a.e. \quad [d\Phi_{C_3}] \quad (85)$$

*Proof.* We first remark that we have not indicated the dependence of  $C_0 = -\frac{1}{2\pi} \log(m_0 d(x, y))$  on the underlying surface, because when there is an ambiguity, the metrics are the same. The proposition follows from the definition (56) for  $C_0$ -regularization, and Remark 5.  $\square$

**Corollary 1.**

$$\int_{\phi_1} \mathcal{Z}_1(\Sigma_2)(\phi_2, \phi_1) \mathcal{Z}_1(\Sigma_1)(\phi_1) \quad (86)$$

$$= \int_{\phi_1} F_P(\phi_2, \phi_1) ((S_1 \sqcup S_2)_*(d\hat{\psi}_{C_2}))^{1/2}(\phi_1, \phi_2) (S_{1*}(d\hat{\phi}_{C_1}))^{1/2}(\phi_1) \quad (87)$$

where

$$F_P(\phi_2, \phi_1) = \int e^{-\int_{\Sigma_3} :P(\Phi):_{C_0}} [d\Phi_{C_3} | \Phi_{S_1} = \phi_1, \Phi_{S_2} = \phi_2] \quad (88)$$

*Proof.* Proposition 2, applied to (83), implies that (86) equals

$$= \int_{\phi_1} \left( \int e^{-\int_{\Sigma_3} P(\Phi):C_0} [d\phi_{C_1} | \phi_{S_1} = \phi_1] \times [d\psi_{C_2} | \psi_{S_1} = \phi_1, \psi_{S_2} = \phi_2] \right) \quad (89)$$

$$((S_1 \sqcup S_2)_*(d\hat{\psi}_{C_2}))^{1/2}(\phi_2, \phi_1)(S_{1*}(d\hat{\phi}_{C_1}))^{1/2}(\phi_1) \quad (90)$$

By (81)  $d\Phi_{C_3}$  is obtained by (normalized) conditioning  $d\phi_{C_1} \times d\psi_{C_2}$  so that  $(\phi_s)_{S_1} = (\psi_s)_{S_1}$ . Thus

$$[d\phi_{C_1} | \phi_{S_1} = \phi_1] \times [d\psi_{C_2} | \psi_{S_1} = \phi_1, \psi_{S_2} = \phi_2] \quad (91)$$

$$= [d\Phi_{C_3} | \Phi_{S_1} = \phi_1, \Phi_{S_2} = \phi_2] \quad (92)$$

Inserting this into (90) yields the proposition.  $\square$

*5.3. Parts 3 and 4. Sewing of normalized background measures and  $\zeta$ -regularized volumes.* Now we need to compare the expression in Corollary 1 with  $\mathcal{Z}_1(\Sigma_3)$ . By Lemma 5

$$\mathcal{Z}_1(\Sigma_3) = \int e^{-\int_{\Sigma_3} P(\Phi):C_0} [d\Phi_{C_3} | \Phi_{S_2} = \phi_2] ((S_2)_*(d\hat{\Phi}_{C_3}))^{1/2}(\phi_2). \quad (93)$$

$$= \int F_P(\phi_2, \phi_1) ([d\Phi_{C_3}^s : \Phi_{S_2}^s = \phi_2]) ((S_2)_*(d\hat{\Phi}_{C_3}))^{1/2}(\phi_2) \quad (94)$$

where  $F_P$  is as in Corollary 1.

To complete the proof of the Theorem, in comparing (86) and (94), it is clear that we need to compare the measures (with values in half-densities) in the two integrals. These measures do not depend upon  $P$  (all the  $P$ -dependence is in  $F_P$ ).

**Proposition 3.** *Suppose that  $P = 0$ . For a.e.  $\phi_2$ , the following equality of measures on fields  $\phi_1$  holds:*

$$\mathcal{Z}(\Sigma_2)(\phi_2, \phi_1) \mathcal{Z}(\Sigma_1)(\phi_1) = \quad (95)$$

$$[d\Phi_{C_3}^s | \Phi_{S_2}^s = \phi_2](\phi_1) \mathcal{Z}(\Sigma_3)(\phi_2) \quad (96)$$

*Remark 7.* (a) The measures involved in this statement are Gaussian, hence eminently computable. The nontrivial content of the statement involves understanding the way in which  $\zeta$ -determinants mesh with the determinants which arise in calculating compositions of half-densities.

(b) Since  $[d\Phi_{C_3}^s | \Phi_{S_2}^s = \phi_2]$  is a probability measure, the free version of the Theorem follows from this proposition by integrating 1 on both sides:

$$\int_{\phi_1} \mathcal{Z}(\Sigma_2)(\phi_2, \phi_1) \mathcal{Z}(\Sigma_1)(\phi_1) = \mathcal{Z}(\Sigma_3)(\phi_2). \quad (97)$$

At the projective level, this equality has an important interpretation in terms of the composition of Lagrangian subspaces (see page 147 of [10] or [16] for



the general definitions). Given a 1-manifold  $S$ , let  $Q(S)$  denote 'position space'  $W^{1/2}(S)$ . Then as Lagrangian subspaces, the composition of

$$W_0^1(\Sigma_1, m)^\perp = \left\{ \begin{pmatrix} \phi_1 \\ D_{\Sigma_1} \phi_1 \end{pmatrix} \right\} \subset T^*Q(S_1) = Q(S_1) \oplus Q(S_1)^* \quad (98)$$

(the Cameron-Martin space of  $\mathcal{Z}_1(\Sigma_1)^2$ , and Helmholtz solution space on  $\Sigma_1$ ) with

$$W_0^1(\Sigma_2, m)^\perp = \left\{ \begin{pmatrix} \phi_2 \\ A\phi_2 + B\phi_1 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ -(B^t\phi_2 + D\phi_1) \end{pmatrix} \right\} \quad (99)$$

$$\subset T^*Q(S_2) \times T^*Q(S_1) \quad (100)$$

(the Cameron-Martin space of  $\mathcal{Z}_1(\Sigma_2)^2$ , and Helmholtz solution space on  $\Sigma_2$ , where  $D_{\Sigma_2}$  has been written as a  $2 \times 2$  matrix, as in (108) below, and the minus sign has been inserted because the intrinsic orientation of  $S_1$  is opposite the  $\Sigma_2$ -induced orientation (see (21)), is

$$W_0^1(\Sigma_2 \circ \Sigma_1, m)^\perp = \left\{ \begin{pmatrix} \phi_2 \\ D_{\Sigma_3} \phi_2 \end{pmatrix} \right\} \subset T^*Q(S_2) \quad (101)$$

(the Cameron-Martin space of  $\mathcal{Z}_1(\Sigma_3)^2$ , and Helmholtz solution space on  $\Sigma_3$ ).

*Proof.* In the course of the proof, we will apply Theorem B of [3] a number of times. In applying this theorem, when we consider the Laplacian  $\Delta_{\Sigma_i}$ , it will be understood that we are imposing a Dirichlet boundary condition.

Reflecting the decomposition (73), Theorem B of [3] implies that

$$\det_\zeta(m_0^2 + \Delta_{\hat{\Sigma}_i}) = \det_\zeta(m_0^2 + \Delta_{\Sigma_i}) \det_\zeta(2D_{\Sigma_i}) \det_\zeta(m_0^2 + \Delta_{\Sigma_i}^*) \quad (102)$$

$$= \det_\zeta(m_0^2 + \Delta_{\Sigma_i})^2 \det_\zeta(2D_{\Sigma_i}). \quad (103)$$

Reflecting the decomposition (81), a slightly extended version of Theorem B implies that

$$\det_\zeta(m_0^2 + \Delta_{\Sigma_3}) = \det_\zeta(m_0^2 + \Delta_{\Sigma_1}) \det_\zeta(m_0^2 + \Delta_{\Sigma_2}) \det_\zeta(D_{\Sigma_1, \Sigma_2}) \quad (104)$$

where  $D_{\Sigma_1, \Sigma_2}$  is the pseudodifferential operator on  $S_1$  which has an inverse with kernel  $(m_0^2 + \Delta_{\Sigma_3})^{-1}$ .

The statement of the proposition involves half-densities, in the variable  $\phi_2$ . To prove the proposition, it suffices to show that

$$\mathcal{Z}(\Sigma_1^*)(\phi_1^*) \mathcal{Z}(\Sigma_2^*)(\phi_1^*, \phi_2) \mathcal{Z}(\Sigma_2)(\phi_2, \phi_1) \mathcal{Z}(\Sigma_1)(\phi_1) = \quad (105)$$

$$[d\Phi_{C_3}^s | \Phi_{S_2}^s = \phi_2](\phi_1) \mathcal{Z}(\Sigma_3)^2(\phi_2) [d\Phi_{C_3}^{*s} | \Phi_{S_2}^{*s} = \phi_2](\phi_1^*), \quad (106)$$

as measures on random fields  $\phi_1$ ,  $\phi_2$ , and  $\phi_1^*$ . To clarify the notation involved in the statement, there is an underlying factorization

$$\hat{\Sigma}_3 = \Sigma_1^* \circ \Sigma_2^* \circ \Sigma_2 \circ \Sigma_1, \quad (107)$$

and  $\phi_1$  is a random field on  $S_1$ , the outgoing boundary of  $\Sigma_1$ ,  $\phi_2$  is a random field on  $S_2$ , the outgoing boundary of  $\Sigma_2$ , and  $\phi_1^*$  is a random field on  $S_1^*$ , the outgoing

boundary of  $\Sigma_2^*$ . To prove (106), we will compute the Fourier transforms of both sides.

Our strategy of proof will involve first doing some intermediate calculations heuristically (which should serve the dual purpose of illuminating the meaning of the statements), and then justifying the answers (by noting that the calculations are valid in finite dimensions, and taking limits).

We will write  $D_{\Sigma_2}$  as a  $2 \times 2$  matrix,

$$D_{\Sigma_2} = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \quad (108)$$

relative to the coordinates  $(\phi_2, \phi_1)$ . Thus for example  $D$  has the following meaning: given  $\phi_1$ , calculate the Helmholtz solution on  $\text{int}(\Sigma_2)$  which has boundary value  $\phi_1$  on  $S_1$  and vanishing boundary value on  $S_2$ ; then  $D\phi_1$  is the inward (from the perspective of  $\Sigma_2$ ) normal derivative along  $S_1$ . We will use the two identities:

$$D_{\Sigma_1} + D = D_{\Sigma_1, \Sigma_2} \quad (109)$$

$$A - B(D_{\Sigma_1} + D)^{-1}B^t = D_{\Sigma_3} \quad (110)$$

The first is straightforward. The second is a coordinate expression of (b) of Remark 7, because (110) is equivalent to

$$D_{\Sigma_3}\phi_2 = A\phi_2 + B\phi_1, \quad -(B^t\phi_2 + D\phi_1) = D_{\Sigma_1}\phi_1. \quad (111)$$

We will similarly write  $D_{\Sigma_2^*}$ , in terms of  $A^*$ ,  $B^*$ , and  $D^*$ .

In the calculations which follow, we will, in intermediate heuristic steps, use matrix notation for various pairings. For example the probability measure  $\mathcal{Z}_1(\Sigma_1)^2$  will be represented by the heuristic expression

$$\det(2D_{\Sigma_1})^{1/2} e^{-\frac{1}{2}\phi_1^t(2D_{\Sigma_1})\phi_1} d\phi_1. \quad (112)$$

We will also use the identity (valid in finite dimensions)

$$\det(2D_{\Sigma_2}) = \det(2D)\det(2(A - BD^{-1}B^t)), \quad (113)$$

which follows from the factorization

$$2D_{\Sigma_2} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2(A - BD^{-1}B^t) & 0 \\ 0 & 2D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}B^t & 1 \end{pmatrix}. \quad (114)$$

We first calculate the Gaussian integral

$$\int_{\phi_1} e^{-i(f_1, \phi_1)} \mathcal{Z}_1(\Sigma_2)(\phi_2, \phi_1) \mathcal{Z}_1(\Sigma_1)(\phi_1) = \quad (115)$$

$$\int_{\phi_1} e^{-i(f_1, \phi_1)} \det(2D_{\Sigma_2})^{1/4} e^{-\frac{1}{2}(\phi_2, \phi_1^t) \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix}} (d\phi_1 d\phi_2)^{1/2} \quad (116)$$

$$\det(2D_{\Sigma_1})^{1/4} e^{-\frac{1}{2}\phi_1^t D_{\Sigma_1} \phi_1} (d\phi_1)^{1/2} \quad (117)$$

$$= \det(2D_{\Sigma_2})^{1/4} \det(2D_{\Sigma_1})^{1/4} \quad (118)$$

$$\int_{\phi_1} \exp(-\frac{1}{2}(\sqrt{D_{\Sigma_1} + D}\phi_1 + \sqrt{D_{\Sigma_1} + D}^{-1}(B^t\phi_2 + if_1))^t) \quad (119)$$

$$(\sqrt{D_{\Sigma_1} + D}\phi_1 + \sqrt{D_{\Sigma_1} + D}^{-1}(B^t\phi_2 + if_1))d\phi_1 \times \quad (120)$$

$$\exp(\frac{1}{2}(B^t\phi_2 + if_1)^t(D_{\Sigma_1} + D)^{-1}(B^t\phi_2 + if_1))\exp(-\frac{1}{2}\phi_2^t A\phi_2)(d\phi_2)^{1/2} \quad (121)$$

$$= \det(2D_2 D_{\Sigma_1}(D_{\Sigma_1} + D)^{-2})^{1/4} \det(2(A - BD^{-1}B^t))^{1/4} \quad (122)$$

$$e^{-\frac{1}{2}f_1^t(D_{\Sigma_1} + D)^{-1}f_1} e^{i\phi_2^t B(D_{\Sigma_1} + D)^{-1}f_1} \quad (123)$$

$$e^{-\frac{1}{2}\phi_2^t \{A - B(D_{\Sigma_1} + D)^{-1}B^t\}\phi_2} (d\phi_2)^{1/2} \quad (124)$$

(we also used (113) in the last step).

We now calculate, in terms of the identities (109)-(110) (and using reflection symmetry), that

$$\int e^{-i((f_1, \phi_1) + (f_2, \phi_2) + (f_1^*, \phi_1^*))} \quad (125)$$

$$\mathcal{Z}_1(\Sigma_1^*)(\phi_1^*) \mathcal{Z}_1(\Sigma_2^*)(\phi_1^*, \phi_2) \mathcal{Z}_1(\Sigma_2)(\phi_2, \phi_1) \mathcal{Z}_1(\Sigma_1)(\phi_1) \quad (126)$$

$$= \det(4D(D_{\Sigma_1} + D)^{-1}D_{\Sigma_1}(D_{\Sigma_1} + D)^{-1})^{1/2} \det(2(A - BD^{-1}B^t))^{1/2} \quad (127)$$

$$\int e^{-\frac{1}{2}(f_1^t D_{\Sigma_1}^{-1} f_1 + f_1^* D_{\Sigma_1^*}^{-1} f_1^* + f_2^t D_{\Sigma_2}^{-1} f_2 + f_2^* D_{\Sigma_2^*}^{-1} f_2^*)} e^{i\phi_2^t (BD_{\Sigma_1}^{-1} f_1 + B^* D_{\Sigma_1^*}^{-1} f_1^* - f_2)} \quad (128)$$

$$e^{-\frac{1}{2}\phi_2^t 2D_{\Sigma_3}\phi_2} d\phi_2 \quad (129)$$

$$= \det(4(1 + D_{\Sigma_1}^{-1}D)^{-1}(1 + D^{-1}D_{\Sigma_1})^{-1})^{1/2} \det(2(A - BD^{-1}B^t)(2D_{\Sigma_3})^{-1})^{1/2} \quad (130)$$

$$e^{-\frac{1}{2}(f_1, (m_0^2 + \Delta_{\Sigma_3})^{-1}f_1) + (f_1^*, (m_0^2 + \Delta_{\Sigma_3^*})^{-1}f_1^*)} \quad (131)$$

$$e^{-\frac{1}{2}(B(m_0^2 + \Delta_{\Sigma_3})^{-1}f_1 + B^*(m_0^2 + \Delta_{\Sigma_3^*})^{-1}f_1^* - f_2), (m_0^2 + \Delta_{\Sigma_3})^{-1}(B(m_0^2 + \Delta_{\Sigma_3})^{-1}f_1 + B^*(m_0^2 + \Delta_{\Sigma_3^*})^{-1}f_1^* - f_2)} \quad (132)$$

The expression we have obtained for the Fourier transform is correct for the following reasons. Our intermediate calculations are valid provided that all the objects involved are understood to be finite dimensional. In particular we can consider compatible compressions of the positive operators  $D_{\Sigma_1}$ ,  $D_{\Sigma_2}$ ,  $D_{\Sigma_2^*}$ , and  $D_{\Sigma_1^*}$ . For example we can consider the positive operators  $pD_{\Sigma_1}p$ ,  $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} D_{\Sigma_2} \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ ,  $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} D_{\Sigma_2^*} \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ , and  $pD_{\Sigma_1^*}p$ , where  $p$  is the projection corresponding to a bounded portion of the spectrum of  $D_{\Sigma_1}$  (where  $D_{\Sigma_2}$  is written as in (108)). As the cutoff  $p$  is removed, the Gaussian measure corresponding to  $pD_{\Sigma_1}p$  will converge weakly to  $\mathcal{Z}_1(\Sigma_1)^2$  (the Gaussian corresponding to  $D_{\Sigma_1}$ ), and so on. We also observe that

$$\frac{1}{4}(2 + D^{-1}D_{\Sigma_1} + D_{\Sigma_1}^{-1}D) \quad (133)$$

and

$$(A - BD^{-1}B^t)^{-1}(A - B(D_{\Sigma_1} + D)^{-1}B^t) \quad (134)$$

$$= (1 - A^{-1}BD^{-1}B^t)^{-1}(1 - A^{-1}B(1 + D^{-1}D_{\Sigma_1})^{-1}D^{-1}B^t) \quad (135)$$

are of the form  $1 + T$ , where  $T$  is trace class. This is true of (133), because  $D^{-1}D_{\Sigma_1} = 1 + H$ , where  $H$  is Hilbert-Schmidt, hence

$$D^{-1}D_{\Sigma_1} + (D^{-1}D_{\Sigma_1})^{-1} = 2 + T, \quad T = (1 + H)^{-1}H^2, \quad (136)$$

and  $T$  is trace class (the fact is that  $D - D_{\Sigma_1}$  is a smoothing operator, so that  $H$  itself is trace class; this follows from use of (55)). This is true for (135), because  $A^{-1}B$  and  $D^{-1}B^t$  are smoothing operators. These considerations imply that the determinants in the last line of (132) are well-defined. Furthermore, if we insert the cutoff  $p$ , the corresponding determinants will converge, as  $p \rightarrow 1$ . This implies that we can take a limit of finite dimensional approximations to justify our formula for the Fourier transform (126).

We now claim that the Fourier transform of the left hand side of (106)

$$= \det(m_0^2 + \Delta_{\hat{\Sigma}_3})^{-1/2} e^{-\frac{1}{2}(f_1, (m_0^2 + \Delta_{\Sigma_3})^{-1}f_1) + (f_1^*, (m_0^2 + \Delta_{\Sigma_3}^*)^{-1}f_1^*)} \quad (137)$$

$$e^{-\frac{1}{2}(B(m_0^2 + \Delta_{\Sigma_3})^{-1}f_1 + B^*(m_0^2 + \Delta_{\Sigma_3}^*)^{-1}f_1^* - f_2), (m_0^2 + \Delta_{\hat{\Sigma}_3})^{-1}(B(m_0^2 + \Delta_{\Sigma_3})^{-1}f_1 + B^*(m_0^2 + \Delta_{\Sigma_3}^*)^{-1}f_1^* - f_2)} \quad (138)$$

To justify this claim, we need to show

$$\det(2D_{\Sigma_1}2D(D_{\Sigma_1} + D)^{-2})^{1/2} \det(2(A - BD^{-1}B^t)(2D_{\Sigma_3})^{-1})^{1/2} \quad (139)$$

$$\det_{\zeta}(m_0^2 + \Delta_{\hat{\Sigma}_2})^{-1/2} \det_{\zeta}(m_0^2 + \Delta_{\hat{\Sigma}_1})^{-1/2} = \det_{\zeta}(m_0^2 + \Delta_{\hat{\Sigma}_3})^{-1/2} \quad (140)$$

Using (4.32) and (4.33), this is equivalent to

$$\det(2D_{\Sigma_1}2DD_{\Sigma_1, \Sigma_2}^{-2})^{1/2} \det(2(A - BD^{-1}B^t)(2D_{\Sigma_3})^{-1})^{1/2} \quad (141)$$

$$\det_{\zeta}(2D_{\Sigma_1})^{-1/2} \det_{\zeta}(2D_{\Sigma_2})^{-1/2} \det_{\zeta}(D_{\Sigma_1, \Sigma_2}) \det_{\zeta}(2D_{\Sigma_3})^{1/2} = 1 \quad (142)$$

To simplify this, we will use the well-known fact that  $\det_{\zeta}(AB) = \det_{\zeta}(A)\det(B)$ , when  $B = 1 + T$ ,  $T$  trace class (see [8] or [12]). This implies that (142) is equivalent to

$$\det_{\zeta}(2D_{\Sigma_1}2D)^{1/2} \det_{\zeta}(2(A - BD^{-1}B^t))^{1/2} \quad (143)$$

$$\det_{\zeta}(2D_{\Sigma_1})^{-1/2} \det_{\zeta}(2D_{\Sigma_2})^{-1/2} = 1 \quad (144)$$

Together with the factorization following (113), this also implies that

$$\det_{\zeta}(2D_{\Sigma_2}) = \det_{\zeta}(2D) \det_{\zeta}(2(A - BD^{-1}B^t)). \quad (145)$$

Thus (146) is equivalent to showing that the multiplicative anomaly

$$F(2D_{\Sigma_1}, 2D) = \det_{\zeta}(2D_{\Sigma_1}2D)^{1/2} \det_{\zeta}(2D_{\Sigma_1})^{-1/2} \det_{\zeta}(2D)^{-1/2} = 1 \quad (146)$$

It is well-known that this vanishes, because  $D - D_{\Sigma_1}$  is a smoothing operator (see [8] or [12]).

We have now established that (138) is an expression for the Fourier transform of the left hand side of (106).

We will now calculate the Fourier transform of the right hand side of (106), along the same lines. As we did for  $D_{\Sigma_2}$ , we will write  $D_{\Sigma_1, \Sigma_2}$  as a  $2 \times 2$  matrix

$$D_{\Sigma_1, \Sigma_2} = \begin{pmatrix} \alpha & \beta \\ \beta^t & \delta \end{pmatrix} \quad (147)$$

relative to the coordinates  $(\phi_2, \phi_1)$ . The crucial fact is that  $B = \beta$ .

We first calculate (heuristically)

$$\int_{\phi_1} e^{-i(f_1, \phi_1)} [d\Phi_{C_3}^s | \Phi_{S_2} = \phi_2] \mathcal{Z}_1(\Sigma_3)(\phi_2) \quad (148)$$

$$= \int_{\phi_1} e^{-i(f_1, \phi_1)} \det(\delta)^{1/2} e^{\frac{1}{2}\phi_2^t(\alpha - \beta\delta^{-1}\beta^t)\phi_2} \quad (149)$$

$$e^{-\frac{1}{2}(\phi_2^t, \phi_1^t) \begin{pmatrix} \alpha & \beta \\ \beta^t & \delta \end{pmatrix} \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix}} d\phi_1 \det(2D_{\Sigma_3})^{1/4} e^{-\frac{1}{2}\phi_2^t D_{\Sigma_3} \phi_2} (d\phi_2)^{1/2} \quad (150)$$

$$= \det(2D_{\Sigma_3})^{1/4} e^{-\frac{1}{2}f_1^t \delta^{-1} f_1} e^{i\phi_2^t \beta \delta^{-1} f_1} e^{-\frac{1}{2}\phi_2^t D_{\Sigma_3} \phi_2} (d\phi_2)^{1/2} \quad (151)$$

This implies

$$\int e^{-i((f_1, \phi_1) + (f_2, \phi_2) + (f_1^*, \phi_1^*))} \quad (152)$$

$$[d\Phi_{C_3}^s | \Phi_{S_2}^s = \phi_2](\phi_1) \mathcal{Z}_1(\Sigma_3)^2(\phi_2) [d\Phi_{C_3}^{*s} | \Phi_{S_2}^{*s} = \phi_2](\phi_1^*) = \quad (153)$$

$$e^{-\frac{1}{2}(f_1, (m_0^2 + \Delta_{\Sigma_3})^{-1} f_1) + (f_1^*, (m_0^2 + \Delta_{\Sigma_3}^*)^{-1} f_1^*)} \quad (154)$$

$$e^{-\frac{1}{2}(\beta(m_0^2 + \Delta_{\Sigma_3})^{-1} f_1 - \beta^*(m_0^2 + \Delta_{\Sigma_3}^*)^{-1} f_1^* - f_2, (m_0^2 + \Delta_{\Sigma_3})^{-1} f_1 - \beta^*(m_0^2 + \Delta_{\Sigma_3}^*)^{-1} f_1^* - f_2)} \quad (155)$$

This last equation is justified, by noting that the calculations leading to it are valid in finite dimensions and taking limits.

Using  $B = \beta$ , it is now clear that the Fourier transform of the right hand side of (106) equals (138). This proves (106), and completes the proof of the proposition.  $\square$

As we remarked above, this proves part (a) of the Theorem, assuming that the incoming boundary of  $\Sigma_1$  is empty and the outgoing boundary of  $\Sigma_2$  is nonempty. The proofs in the other two cases for (a) involve straightforward modifications.

To prove (b), suppose that  $\Sigma = \Sigma_1 \circ \Sigma_2$ . Then

$$\text{trace}(\mathcal{Z}(\Sigma)) = \mathcal{Z}(|\Sigma_1|^*) \circ \mathcal{Z}(|\Sigma_2|) = \mathcal{Z}(|\Sigma_1|^* \circ |\Sigma_2|) = \mathcal{Z}(\hat{\Sigma}). \quad (156)$$

$$(157)$$

This completes the proof of the Theorem.

## 6. Appendix A: Half-Densities

Suppose that  $X$  is a standard Borel space, and  $\mathcal{C}$  is a measure class on  $X$ . Let  $\bar{\mathcal{C}}$  denote the union of all measure classes which are absolutely continuous with respect to  $\mathcal{C}$ , and let  $\bar{\mathcal{C}}_f$  denote the subset of finite measures. There is a real separable Hilbert space,  $\mathcal{H}(\mathcal{C})$ , the space of half-densities relative to  $\mathcal{C}$ , and a bilinear map

$$\mathcal{H}(\mathcal{C}) \times \mathcal{H}(\mathcal{C}) \rightarrow \bar{\mathcal{C}}_f, \quad (158)$$

which are canonically associated to  $\mathcal{C}$ . We will define the space of half densities in terms of its representations.

Fix a positive representative  $\nu$  for  $\mathcal{C}$ . There is an isomorphism of Hilbert spaces

$$L^2(X, \nu; \mathbb{R}) \rightarrow \mathcal{H}(\mathcal{C}) : f \rightarrow f(d\nu)^{1/2}, \quad (159)$$

and in terms of this isomorphism, the map (158) is given by

$$f(d\nu)^{1/2} \otimes g(d\nu)^{1/2} \rightarrow fg d\nu \quad (160)$$

If one chooses another positive representative for  $\mathcal{C}$ , say  $\mu$ , then

$$f(d\nu)^{1/2} = h(d\mu)^{1/2} \Leftrightarrow f = h\left(\frac{d\mu}{d\nu}\right)^{1/2} \quad (161)$$

where  $f \in L^2(d\nu)$ ,  $h \in L^2(d\mu)$ , and  $(\frac{d\mu}{d\nu})^{1/2}$  denotes the positive square root of this positive function. In an obvious way, these identifications can be used to give a formal definition of  $\mathcal{H}(\mathcal{C})$ .

We now list a number of elementary facts about spaces of half-densities.

(1) If  $\mathcal{C}_1 \ll \mathcal{C}_2$ , then there is a canonical isometric embedding

$$\mathcal{H}(\mathcal{C}_1) \rightarrow \mathcal{H}(\mathcal{C}_2) : f(d\nu_1)^{1/2} \rightarrow f\left(\frac{d\nu_1}{d\nu_2}\right)^{1/2}(d\nu_2)^{1/2}, \quad (162)$$

where  $\nu_i$  is a positive representative for  $\mathcal{C}_i$ .

(2) The isomorphism (159), and the coordinate transformation (162), show that there is a distinguished positive cone inside  $\mathcal{H}(\mathcal{C})$ , corresponding to non-negative functions in (159). We will denote this cone by  $\mathcal{H}(\mathcal{C})^+$ . Given a positive finite measure,  $\nu$ ,  $\nu^{1/2}$  will denote the positive square root in the space of half densities.

(3) The natural representation of  $L^\infty(\mathcal{C})$  by multiplication operators on  $L^2(X, \nu)$  corresponds to a well-defined natural action

$$L^\infty(\mathcal{C}) \times \mathcal{H}(\mathcal{C}) \rightarrow \mathcal{H}(\mathcal{C}) : F \otimes \delta \rightarrow F \cdot \delta \quad (163)$$

Conversely given a faithful multiplicity free representation of a commutative Von Neumann algebra

$$\mathcal{A} \times \mathcal{H} \rightarrow \mathcal{H}, \quad (164)$$

there is a measure class  $\mathcal{C}$ , unique up to isomorphism, such that (164) is realized as (163). This is a special case of the spectral theorem (see [6], page 210, theorem 2).

(4) Given disjoint measure spaces  $\mathcal{C}_i$ , there is a canonical isomorphism

$$\mathcal{H}(\mathcal{C}_1) \oplus \mathcal{H}(\mathcal{C}_2) \rightarrow \mathcal{H}(\mathcal{C}_1 \sqcup \mathcal{C}_2) \quad (165)$$

(5) Given a pair of spaces and measure classes  $(X_i, \mathcal{C}_i)$ , there is a measure class  $\mathcal{C}_1 \otimes \mathcal{C}_2$  on  $X_1 \times X_2$  generated by  $\mathcal{C}_1 \times \mathcal{C}_2$ . There is a canonical isomorphism

$$\mathcal{H}(\mathcal{C}_1) \otimes \mathcal{H}(\mathcal{C}_2) \rightarrow \mathcal{H}(\mathcal{C}_1 \otimes \mathcal{C}_2) \quad (166)$$

(6) Suppose that  $\nu$  is a finite positive measure belonging the measure class  $\mathcal{C} \otimes \mathcal{C}$  on  $X \times X$ . Then

$$\nu^{1/2} \in \mathcal{H}(\mathcal{C}) \otimes \mathcal{H}(\mathcal{C}), \quad (167)$$

and hence  $\nu^{1/2}$  can be interpreted as a Hilbert-Schmidt operator on  $\mathcal{H}(\mathcal{C})$ . This operator is positivity-preserving:

$$\nu^{1/2} : \mathcal{H}(\mathcal{C})^+ \rightarrow \mathcal{H}(\mathcal{C})^+ \quad (168)$$

(in [15], page 30, the phrase ‘doubly Markovian map’ is used for this property).

Given  $\nu_1$  and  $\nu_2$ ,

$$\nu_1^{1/2} \circ \nu_2^{1/2} = \nu_3^{1/2}, \quad (169)$$

where  $\nu_3$  is another finite positive measure,

$$\nu_3(\phi, \psi) = \left( \int_{\eta} \nu_1(\phi, \eta)^{1/2} \nu_2(\eta, \psi)^{1/2} \right)^2. \quad (170)$$

This can be summarized as follows.

**Proposition 4.** *The finite positive measures in  $\mathcal{C} \otimes \mathcal{C}$  form a semigroup, with multiplication (169), and this semigroup is represented by positivity-preserving Hilbert-Schmidt operators on  $\mathcal{H}(\mathcal{C})$ .*

(7) Now suppose that the Borel structure on  $X$  is derived from a locally convex linear structure, and  $\mathcal{C}$  is the measure class on  $X$  of a Gaussian measure (chapter 2 of [1]).

**Proposition 5.** *Suppose that  $\nu_1$  and  $\nu_2$  are Gaussian measures belonging to  $\mathcal{C} \otimes \mathcal{C}$ . Then  $\nu_3$ , as in (169)-(170), is a multiple of a Gaussian measure.*

If  $X$  is finite dimensional, this is a consequence of the Theorem in section 3, page 65 of [11]. The Proposition follows in a routine way, after rewriting Howe's formulas to account for normalizations of measures, by taking limits.

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